Should Small and Large Information Economies Have the Same Duration of Copyright?

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Abstract

International copyright harmonization has been pursued without its theoretical foundation being questioned. This paper investigates one possible foundation for such harmonization and asks whether national governments pursuing national self-interests are naturally inclined to choose a uniform term of copyright protection. By developing and analyzing a model of two nations, each with an open creative industry, trading information goods with the other, and setting copyright policy to maximize its national welfare, while taking the other's copyright policy as given. It is found that such nationally self-interested governments generally would choose different terms of copyright protection based on their different creative technologies or different levels of demand for information goods. The foundation for uniform international copyright policy is not a non-cooperative competitive regime.

Introduction

Copyright is the legal underpinning of the increasingly global information economy advanced by digital technologies. It has almost been taken for granted that different nations should have the same copyright protection for information goods. The European Union (EU) has been very active in copyright harmonization and has adopted numerous directives in the area. In justifying the copyright term extension proposed in the U.S. Senate bill S. 483, which later became the U.S. 1998 Sonny Bono Copyright Term Extension Act, the U.S. Senate Report accompanying that bill cited the need to "harmonize" U.S. copyright term with the longer copyright term in EU as a reason for the proposed extension (Senate Report, 1996). Since the bill was enacted into law in 1998, many countries have followed the U.S. to extend their copyright duration to match that of the U.S.

There could be multiple justifications for a uniform international copyright system. This paper investigates perhaps the most fundamental possible theoretical basis of uniform international copyright policy: whether competitive nationally self-interested governments are naturally inclined to adopt uniform copyright policy. We model two countries trading information goods. Each country has a sector of creative industry and consumers of information products. Each country sets its copyright law to maximize the total surplus of its creators and consumers, given the other country's law as given. We analyze the model to see if copyright durations of the two countries be the same.

The model extends current copyright models to a two-economy trading setting. Extant models, e.g., Landes & Posner (1989) and Yuan (2005), assume a single and closed information economy. The model also differs from the conventional trade models of general goods and services because of the unique cost structure of information goods and the need for a copyright policy variable in the model. The model will be analyzed numerically so that it can be more realistic by reflecting several aspects of the international copyright system, such as allowing an open creative section in each country, international trade in information goods, and national policy making for national self-interest.

The main result of the paper is that different competitive nations do not naturally tend to adopt a uniform copyright policy. In particular, a country with larger demand for information goods may be inclined to adopt longer copyright protection to induce larger global creative industries. A country with a smaller demand for information goods may choose to free ride on the creative industries induced by the other and adopt a minimal copyright protection policy. The result differs from that of a single country model, where copyright duration decreases with level of demand for information goods. The result means that a uniform international copyright policy is not based on a non-cooperative competitive regime. The model for a uniform international copyright policy needs to be found elsewhere.

The rest of paper proceeds as follows. The next section develops the two-country trading model. Section 3 simulates the model and presents the results. The paper ends with a brief concluding remark. Some mathematical details leading to the numerical solution are provided in the appendix.

The Model

We model a two country international information economy. In each country, there are creators and consumers of information goods. A creator of a country develops original information products and sells copies of its products to consumers of this and the other country. Creators of one country may face different technological and other business/regulatory environment, therefore, may have different creative cost from the creators of the other country. The two information economies may also differ in their consumers' demand for information goods. The reproduction cost of information goods will be assumed the same across countries to reflect the unique creative cost of information goods.

Each country maximizes its own "national interest" and sets its copyright law to maximize its total surplus of its creators and consumers, taking the copyright law of the other country as given. The law adopted by a country applies to both domestic products and foreign products on the market of the country and is assumed to be enforceable within the border of the country separately from the law of the other country.

The key policy parameter modeled will be the duration of copyright. Duration is a key variable in a copyright system and it may be interpreted more generally as the strength of copyright. Other aspects of copyright will be assumed fixed and uniform across the two countries.

The model will use the following notations:

i_1: index of creators of country 1;
i_2: index of creators of country 2;
n_1: number of creators of country 1;
n_2: number of creators of country 2;
s_{1i}: size of creator i of country 1;
s_{2i}: size of creator i of country 2;
S: total number of first copy products $S=n_1*s_1+n_2*s_2$;
$c_{1i}(s_{1i})$: creative cost of creator i of country 1;
$c_{2i}(s_{2i})$: creative cost of creator i of country 2;
b: per copy reproduction cost of creators of both country 1 and 2;
p_{1i}: price per copy of products of creator i of country 1 ;
p_{2i}: price per copy of products of creator i of country 2;
T_1: copyright duration of country 1;
T_2: copyright duration of country 2;
$D_{11i}(s_{1i}, s_{1-i}, s_{2i}, p_{1i}, p_{1-I}, p_{2i}, t)$: demand for products of creator i of country 1 in country 1;
$D_{12i}(s_{1i}, s_{1-i}, s_{2i}, p_{1i}, p_{1-I}, p_{2i}, t)$: demand for products of creator i of country 1 in country 2;
$D_{21i}(s_{1i}, s_{2-i}, s_{2i}, p_{1i}, p_{2-i}, p_{2i}, t)$: demand for products of creator i of country 2 in country 1;
$D_{22i}(s_{1i}, s_{2-i}, s_{2i}, p_{1i}, p_{2-i}, p_{2i}, t)$: demand for products of creator i of country 2 in country 2;
cs_1: consumer surplus of country 1;
cs_2: consumer surplus of country 2;
w: total consumer surplus, $w \equiv cs_1+cs_2$.

The prices of products of a creator on the domestic and foreign markets are assumed to be the same to avoid the possibility of arbitrage when the products are under copyright protection on both markets. Such arbitrage would be especially easy in the increasingly global digital information economy. When the copyright of a product in a country has expired but has not expired in the other country, an effective importation ban will be assumed.

Profit of creator i of country 1 is:

$$\pi_{1i} = \int_0^{T_1} (d_{11i}(p_{1i} - b)e^{-\gamma t} dt + \int_0^{T_2} (d_{12i}(p_{1i} - b)e^{-\gamma t} dt - c_{1i}(s_{1i}) \quad (1)$$

Profit of creator i of country 2 is:

$$\pi_{2i} = \int_0^{T_1} (d_{21i}(p_{2i} - b)e^{-\gamma t} dt + \int_0^{T_2} (d_{22i}(p_{2i} - b)e^{-\gamma t} dt - c_{2i}(s_{2i}) \quad (2)$$

A creator chooses price and number of information products to create to maximize profit. The first-order conditions are:

$$\frac{\partial \pi_{1i}}{\partial p_{1i}} = 0 \quad (3)$$

$$\frac{\partial \pi_{1i}}{\partial s_{1i}} = 0 \quad (4)$$

$$\frac{\partial \pi_{2i}}{\partial p_{2i}} = 0 \quad (5)$$

$$\frac{\partial \pi_{2i}}{\partial s_{2i}} = 0 \quad (6)$$

A creator also decides whether to enter or stay on the market. The marginal creator makes zero profits. If all creators in a country have the same technology, they will all make zero profit. That is:

$$\pi_{1i} = 0 \quad (7)$$

$$\pi_{2i} = 0 \quad (8)$$

The consumer surplus of country 1 can then be written as:

$$cs_1 = \sum_{i=1}^{n_1} \int_0^{\infty} \left(\int_b^{\infty} d_{11i} dp \right) e^{-\gamma t} dt + \sum_{i=1}^{n_2} \int_0^{\infty} \left(\int_b^{\infty} d_{21i} dp \right) e^{-\gamma t} dt$$

$$- \sum_{i=1}^{n_1} \int_0^{T_1} \left(\int_b^{p_{1i}^*} d_{11i} dp \right) e^{-\gamma t} dt - \sum_{i=1}^{n_2} \int_0^{T_1} \left(\int_b^{p_{2i}^*} d_{21i} dp \right) e^{-\gamma t} dt \quad (9)$$

The consumer surplus of country 2 is:

$$cs_2 = \sum_{i=1}^{n_1} \int_0^{\infty} \left(\int_b^{\infty} d_{12i} dp \right) e^{-\gamma t} dt + \sum_{i=1}^{n_2} \int_0^{\infty} \left(\int_b^{\infty} d_{22i} dp \right) e^{-\gamma t} dt$$

$$- \sum_{i=1}^{n_1} \int_0^{T_2} \left(\int_b^{p_{1i}^*} d_{12i} dp \right) e^{-\gamma t} dt - \sum_{i=1}^{n_2} \int_0^{T_2} \left(\int_b^{p_{2i}^*} d_{22i} dp \right) e^{-\gamma t} dt \quad (10)$$

When all creators in a country are assumed to have the same technology, they all make zero profit and social welfare is the same as consumer surplus.

Country 1's problem is to set duration T_1 to maximize its social welfare cs_1:

$$\frac{dcs_1}{dT_1} = 0 \tag{11}$$

given that (3)-(8) hold and duration T_2 is as set by country 2.

Similarly, country 2's problem to set duration T_2 to maximize *its* social welfare cs_2:

$$\frac{dcs_2}{dT_2} = 0 \tag{12}$$

Given that (3)-(8) hold and T_1 is as set by country 1.

Simulation Results

Solving the above model requires specific forms for the demand and cost functions. Assume the following demand and cost functions:

$$d_{11i} = D_1 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{1i}{}^{-\delta} \prod_{j\neq i} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_2} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} g_1(t) \tag{19}$$

$$d_{12i} = D_2 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{1i}{}^{-\delta} \prod_{j\neq i} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_2} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} g_2(t) \tag{20}$$

$$d_{21i} = D_1 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{2i}{}^{-\delta} \prod_{j\neq i} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_1} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} g_1(t) \tag{21}$$

$$d_{22i} = D_2 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{2i}{}^{-\delta} \prod_{j\neq i} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_1} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} g_2(t) \tag{22}$$

where

$$g_1(t) = \begin{cases} 1 - \dfrac{t}{T_{01}} & if \quad t < T_{01}(1-\theta_1) \\ \theta_1 & otherwise \end{cases} \tag{23}$$

$$g_2(t) = \begin{cases} 1 - \dfrac{t}{T_{02}} & if \quad t < T_{02}(1-\theta_2) \\ \theta_2 & otherwise \end{cases} \tag{24}$$

and

$$c_{1i}(s_{1i}) = c_{01} + a_1 s_{1i}^{\rho_1} \qquad \forall \text{ i of country 1} \tag{25}$$

$$c_{2i}(s_{2i}) = c_{01} + a_2 s_{2i}^{\rho_2} \qquad \forall \text{ i of country 2} \tag{26}$$

where $0<\alpha<1$, $\delta>1$, $\beta>0$, $0\leq \theta_1 <1$, $0\leq \theta_2 <1$, $\rho_1>1$, $\rho_2>1$, and D_1, D_2, T_{01}, T_{02}, $c_{01,}$ c_{02} and a_1 and a_2 are positive constants.

Demand functions (19)-(24) represent that demands in the two countries may differ in level, D_1 and D_2, and long term residual demand, θ_1 and θ_2, and the time it takes for the demands to drop to the residuals, $T_{01}(1-\theta_1)$ and $T_{02}(1-\theta_2)$. Otherwise, all products are treated similarly by domestic and foreign consumers. And the consumers in the two country have the same price elasticity and cross-price elasticity, and preference for variety, as represented by the common values of δ, β, and α, respectively.

Based on the above assumed common price elasticity, it is easy to derive that all creators set a common price for their products in maximizing their individual profits:

$$p_{1i} = p_{2j} = p \equiv \frac{\delta}{\delta-1} b \tag{27}$$

According to cost functions (25) and (26), creators of each country have identical creative costs but may have different creative costs from those of the other country. Possible sources of such cross-country cost difference can be technological and/or regulatory.

Given the identical cost functions within one country, it can be derived that all creators of one country create the same number of first-copy products: $s_{1i} = s_{1j} \equiv s_1$ and $s_{2i} = s_{2j} \equiv s_2$.

We are to solve the model for price p, sizes of creators, s_1 and s_2, number of creators n_1 and n_2, total number of first-copy products S, duration of copyright T_1 and T_2, and consumer surpluses cs_1 and cs_2, and total surplus w. Given the above demand and cost functions, analytical solutions are not found. Numerical solutions can be computed given specific values for the parameters in the functions. Some details of the mathematical steps leading to numerical solutions are given in the appendix.

Assume the following parameter values:

$$[D_1, D_2, \alpha, \delta, \beta, b, T_{01}, T_{02}, \theta_1, \theta_2, \gamma, c_{01}, c_{02.}, a_1, a_2, \rho_1, \rho_2] =$$
$$[10^7\ \ 9{*}10^6\ 0.3\ \ 2\ \ 0.5\ \ 5\ \ 100\ 100\ \ 0.001\ 0.001\ 0.05\ 3{*}10^5\ 3{*}10^5\ 10^4\ 10^4\ 1.2\ 1.2\ ] \tag{28}$$

The following numerical solution is computed:

T_1	T_2	s_1	s_2	S	cs_1	cs_2
9	5	64	64	9627	$1.2 B	$1.2 B

The optimality of the solution is shown in the figure 1. The parameter values represent that consumers of country one has higher level of demand for information goods than those of country two; and the two countries are otherwise the same in consumer demand and creative technology. Given the above parameters, the copyright duration will be 9 years in country one and 5 years in country two; creators in both countries will each create 64 original products; the total number of first-copy products will be 9627; and consumer surplus for both countries will be $1.2 billion. The equality of the welfare of the two countries may be surprising and will be commented below.

Most interestingly, the bigger country chooses a longer copyright protection. The longer duration in a bigger country is further shown in Figure 2 and 3. Figure 2 are obtained by changing the parameter D_1 while fixing D_2 at 10^7 and the other parameters at the values listed in (28). D_1 reflects the level of demand for information good in country one. When D_1 is bigger than D_2, duration in country one is longer than that in country two. As demand level of country one increases, copyright duration of the country increases and that of the other country decreases. This will continue until copyright duration in country one reaches a maximum of 14 years and that of country two reaches a minimum of zero. Similarly, as demand of country one decreases, the copyright duration of the country decreases and that in country two increases. This continues until the duration in country one reaches a minimum of zero and that in country two reaches a maximum of 14.

In another word, the duration of a country increases but that of the other country decreases with consumer demand of this country. This result is the opposite of result of the one country model in Yuan (2005). In the one-country model, copyright duration decreases with the level of demand for information goods as the net result of three individual effects: (a) a higher demand means that more information products should be created and calls for longer protection to induce their creation; (b) a higher demand provides higher profitability for creators during copyright protection and reduces the need for longer copyright protection; (c) a higher demand increases deadweight losses of copyright protection and calls for shorter protection. And the individual effects of (b) and (c) dominate the effect (a) in Yuan (2005).

Why does the duration of a country increase but that of the other country decrease with the consumer demand in this country? And why does it differ from the result of the one-country model? An upward movement in the demand for information goods in this country represents a net increase in creative incentive for creators in the other country. It reduces the need for copyright protection to encourage creation in the other country. Therefore, copyright duration in the other country decreases with demand level in this country. The reduction in copyright duration in the other country, in turn, means a decrease in incentive for creators of this country. The result is an additional individual effect for longer copyright duration in this country. This new individual effect has tipped off the balance of the combined net effect and the net effect now calls for longer duration in this country.

Figure 2 and 3 also show that consumer surpluses in both countries, number of first-copy product per creator, and total number of first-copy products all increase with the level of demand in a country. Note that the second graph of Figure 2 shows that welfare of the two countries are the same when the demand level D_1 is within such interval that country two can respond by changing its copyright duration. In the model, the country with lower level of demand does not necessarily have less power than the larger country. When it can adjust its copyright duration, it will do so to get exactly the same welfare as the larger country. When its duration reaches zero, it loses the ability to respond and obtains lower welfare than the larger country.

Figure 4 and 5 show that, the more slowly the demand for information product dissipates over time, i.e., the longer the economic life of information goods in a country, as represented by larger T_{01} or T_{02}, the longer protection in the country and the shorter duration in the other country.

Figure 6, 7, 8, and 9 show that countries with higher fixed cost of creation and creative cost per product tend to have longer duration of copyright. Figure 10 and 11 show that a country with severer diseconomies of creation tends to have shorter duration of copyright. These results are consistent with result of the one-country model in Yuan (2005).

Conclusion

We have developed and simulated a model of two countries trading information goods, each with an open sector of creative industry, consumers of information goods, a copyright authority which sets copyright policy to maximize its national welfare, while taking the other country's copyright policy as given. The model represents a non-cooperative competitive international copyright policy regime. The model shows that countries with different demand for and creative cost of information goods generally prefer different length of copyright protection. Therefore, the uniform international copyright system pursued by the U.S. and other countries is different from the non-cooperative competitive regime modeled in the paper. The foundation for a uniform international copyright system or international copyright harmonization must be found from other models. These models are left for future research.

Reference

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Senate Report, 1996. Copyright term extension act of 1996, 104th Congress, 2nd Session, Calendar No. 491, July 10. http://www.copyright.gov/legislation/s-rep104-315.html.

Yuan, M. Y. (2005) Does decrease in copying cost support copyright term extension? Information Economics and Policy, 17 (4), 471-494.

Figure 1: Optimality of Equilibrium Policy Solutions

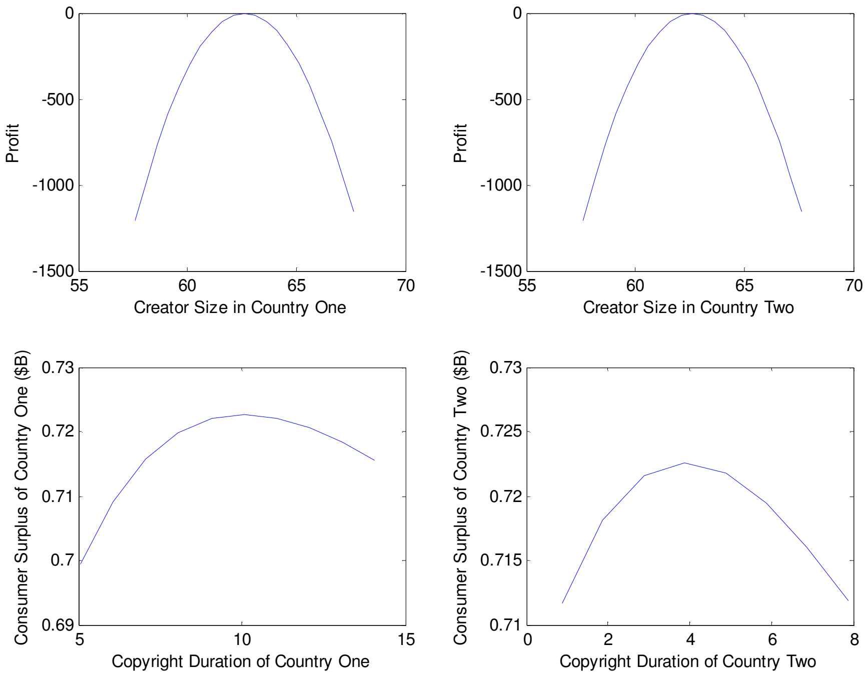


Figure 2. Effect of Consumer Demand of Country one

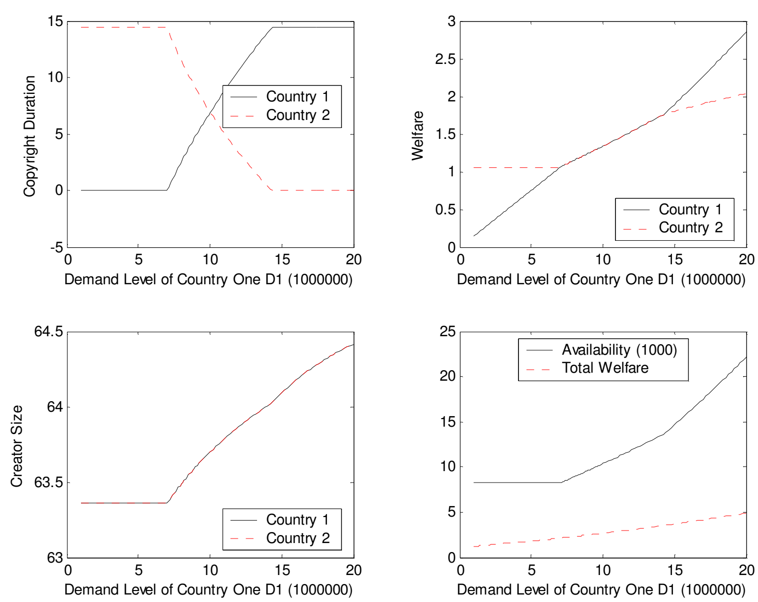


Figure 3. Effect of Consumer Demand of Country Two

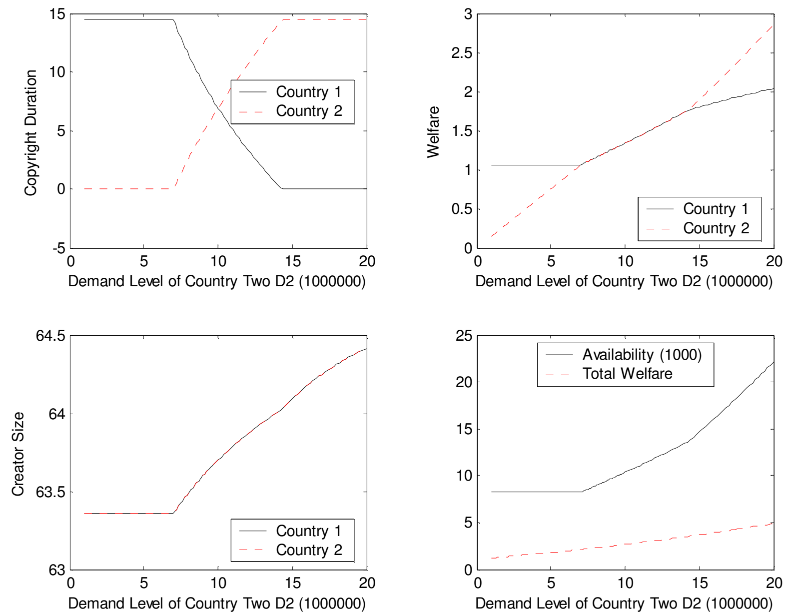


Figure 4. Effect of Product Economic Life in Country 1

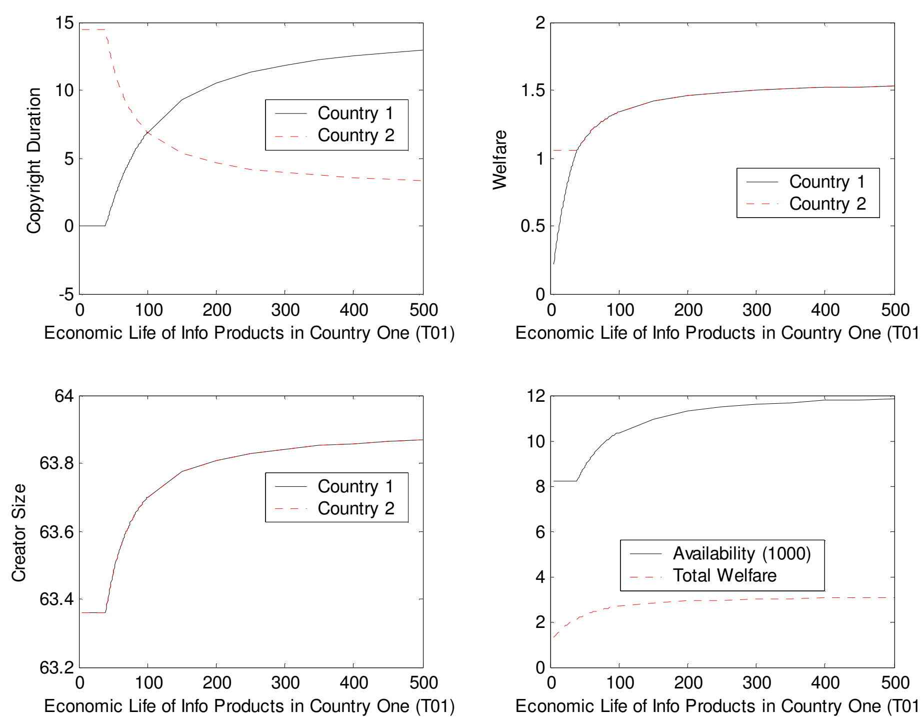


Figure 5. Effect of Product Economic Life in Country 2

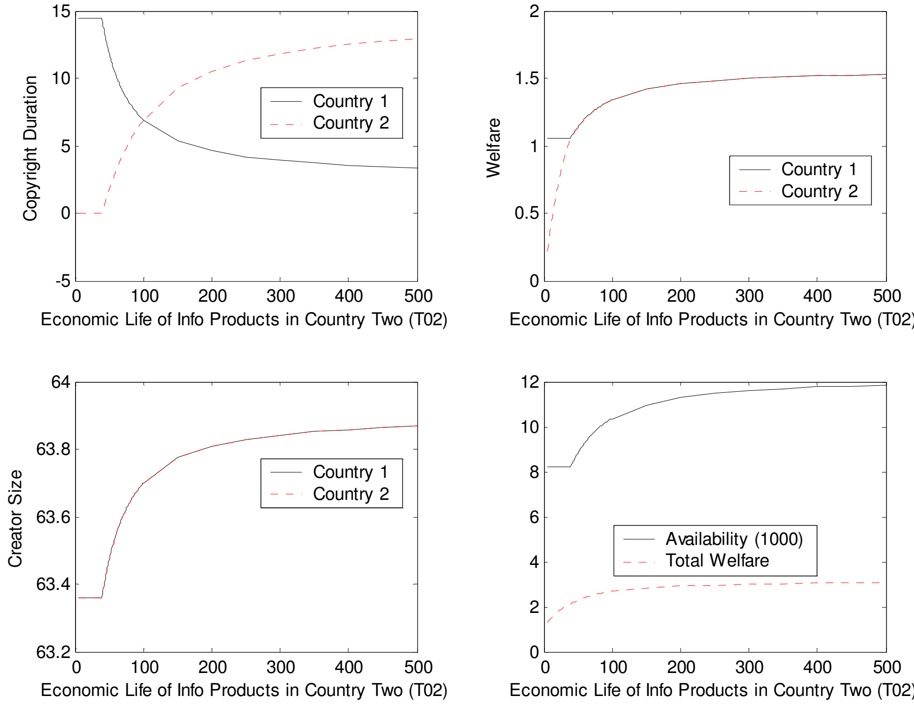


Figure 6. Effect of Fixed Creative Cost in Country 1

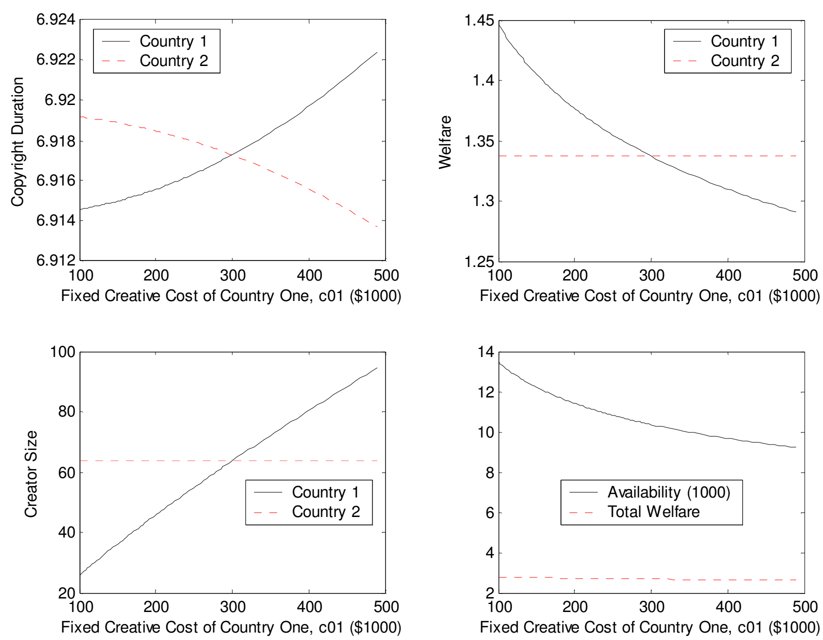


Figure 7. Effect of Fixed Creative Cost in Country 2

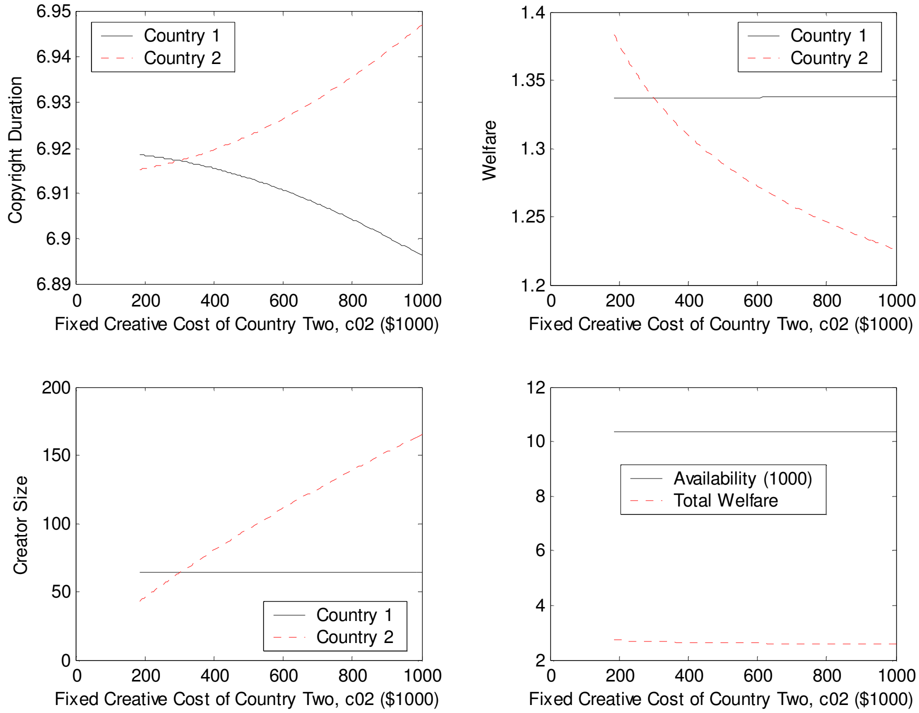


Figure 8. Effect of Per-Product Creative Cost in Country 1

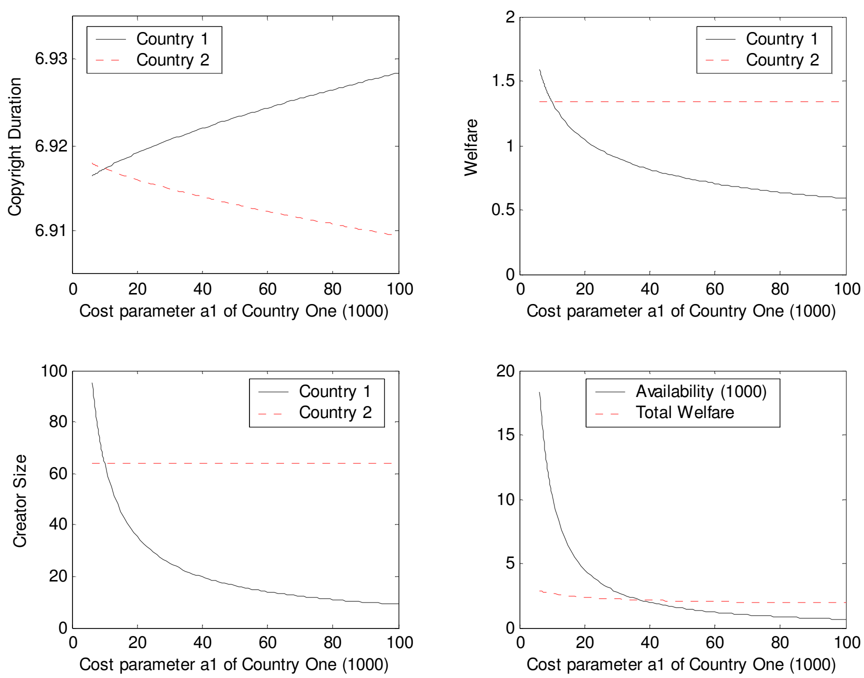


Figure 9. Effect of Per-Product Creative Cost in Country 2

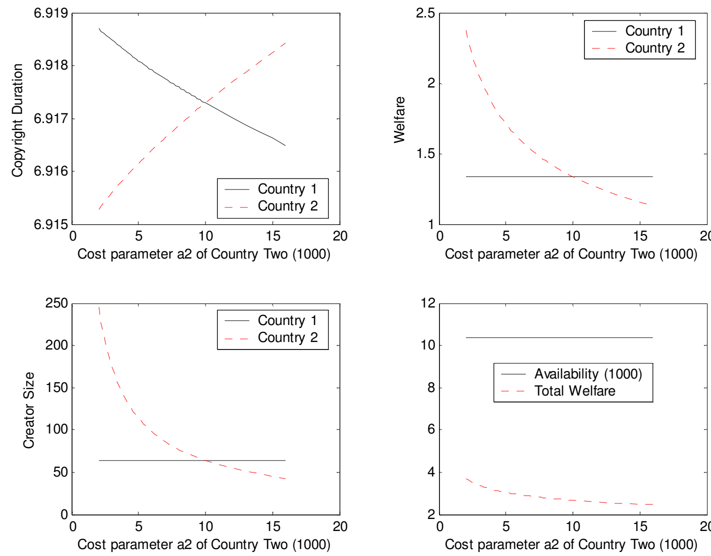


Figure 10. Effect of Diseconomies of Creative Scale in Country 1

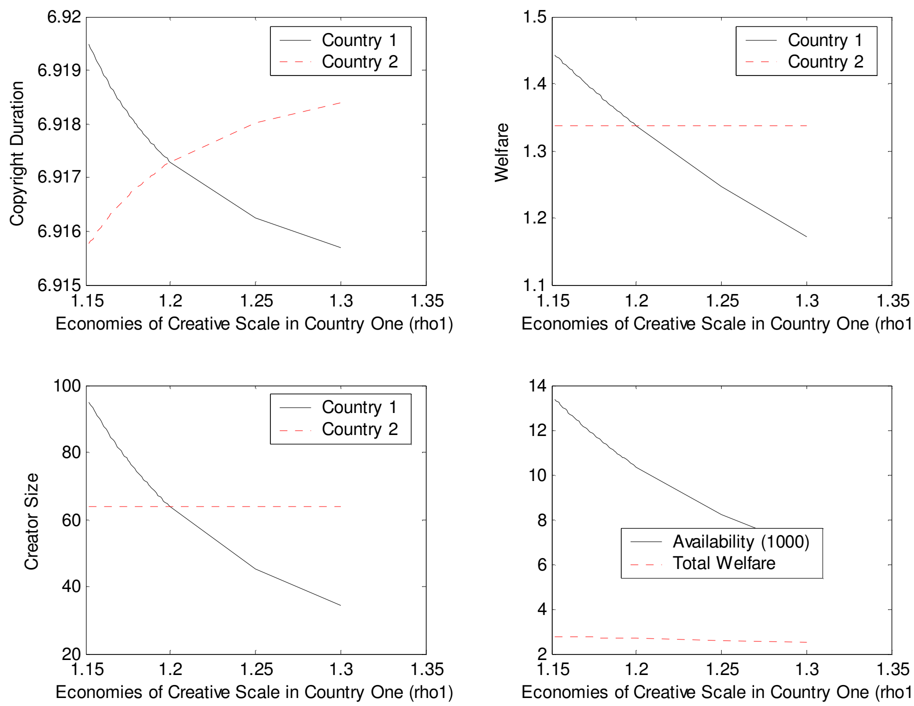
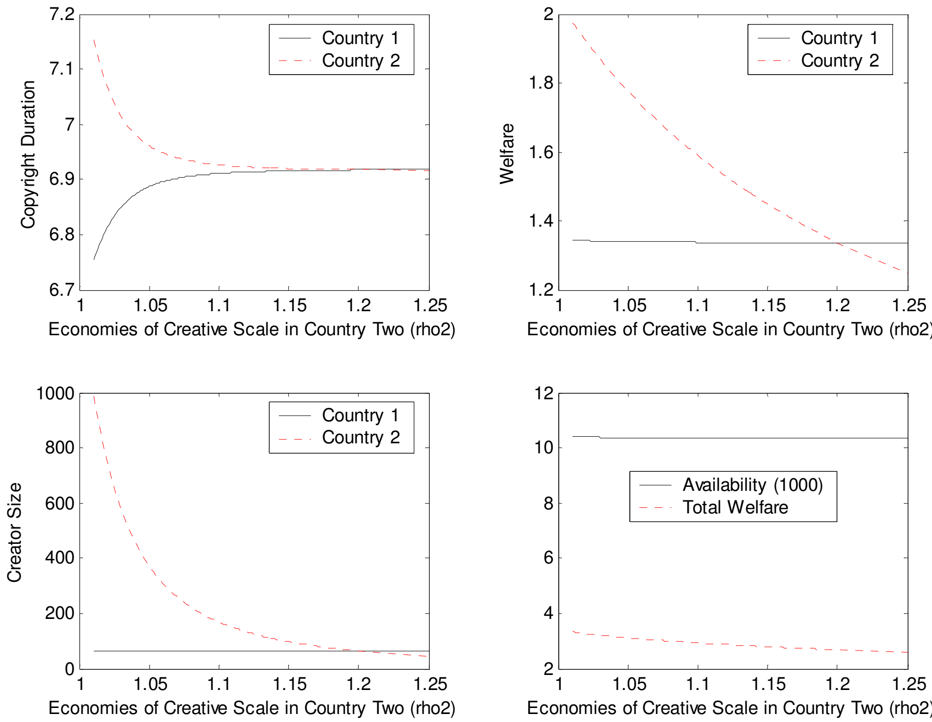


Figure 11. Effect of Diseconomies of Creative Scale in Country 2



Appendix: Mathematical Procedures Leading to Numerical Solution

Copy the demand functions of (19)-(22) from the section of Simulation Results:

$$d_{11i} = D_1 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{1i}{}^{-\delta} \prod_{j\neq i} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_2} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} g_1(t) \tag{A1}$$

$$d_{12i} = D_2 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{1i}{}^{-\delta} \prod_{j\neq i} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_2} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} g_2(t) \tag{A2}$$

$$d_{21i} = D_1 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{2i}{}^{-\delta} \prod_{j\neq i} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_1} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} g_1(t) \tag{A3}$$

$$d_{22i} = D_2 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p_{2i}{}^{-\delta} \prod_{j\neq i} p_{2j}{}^{\frac{\beta}{n_1+n_2-1}} \prod_{j=1}^{n_1} p_{1j}{}^{\frac{\beta}{n_1+n_2-1}} g_2(t) \tag{A4}$$

From pricing decision of (3) and (5), $\frac{\partial \pi_{1i}}{\partial p_{1i}} = 0$ and $\frac{\partial \pi_{2i}}{\partial p_{2i}} = 0$, one has:

$$p_{1i} = p_{2j} = p \equiv \frac{\delta}{\delta-1} b \tag{A5}$$

Plug (A5) into (A1-A4), one has

$$d_{11i} = D_1 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta} g_1(t) \tag{A6}$$

$$d_{12i} = D_2 s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta} g_2(t) \tag{A7}$$

$$d_{21i} = D_1 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta} g_1(t) \tag{A8}$$

$$d_{22i} = D_2 s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta} g_2(t) \tag{A9}$$

Plug (A6-9) into profit function (1) and (2), one has

$$\pi_{1i} = [D_1 G(T_1) + D_2 G(T_2)] s_{1i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta}(p-b) - c_1(s_{1i}) = 0 \tag{A10}$$

$$\pi_{2i} = [D_1 G(T_1) + D_2 G(T_2)] s_{2i} \left(\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}\right)^{\alpha-1} p^{\beta-\delta}(p-b) - c_2(s_{2i}) = 0 \tag{A11}$$

Where

$$G_1(T_1) \equiv \int_0^{T_1} g_1(t)\, e^{-\gamma t} dt \tag{A12}$$

And

$$G_2(T_2) \equiv \int_0^{T_2} g_2(t)\, e^{-\gamma t} dt \tag{A13}$$

From the sizing decisions (4) and (6), $\frac{\partial \pi_{1i}}{\partial s_{1i}} = 0$ and $\frac{\partial \pi_{2i}}{\partial s_{2i}} = 0$, and the marginal profit conditions (7) and (8), $\pi_{1i} = 0$ and $\pi_{2i} = 0$, one has:

$$\frac{c_1(s_{1i})}{s_{1i}} + (\alpha - 1)\frac{c_1(s_{1i})}{\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}} = c_1{}'(s_{1i}) \tag{A14}$$

$$\frac{c_2(s_{2i})}{s_{2i}} + (\alpha - 1)\frac{c_2(s_{2i})}{\sum_{j=1}^{n_1} s_{1j} + \sum_{i=1}^{n_2} s_{2j}} = c_2{}'(s_{2i}) \tag{A15}$$

From identical cost functions (25) and (26) and by symmetry in (A14) and (A15), one has $s_{1i} = s_{1j} \equiv s_1$ and $s_{2i} = s_{2j} \equiv s_2$. Therefore, (A14) and (A15) can be written as:

$$\frac{1}{s_1} + \frac{\alpha-1}{n_1 s_1 + n_2 s_2} = \frac{c_1{}'(s_1)}{c_1(s_1)} \tag{A16}$$

$$\frac{1}{s_2} + \frac{\alpha-1}{n_1 s_1 + n_2 s_2} = \frac{c_2{}'(s_2)}{c_2(s_2)} \tag{A17}$$

(A10) and (A11) become:

$$[D_1 G_1(T_1) + D_2 G_2(T_2)] s_1 (n_1 s_1 + n_2 s_2)^{\alpha-1} p^{\beta-\delta}(p - b) - c_1(s_1) = 0 \tag{A18}$$

$$[D_1 G_1(T_1) + D_2 G_2(T_2)] s_2 (n_1 s_1 + n_2 s_2)^{\alpha-1} p^{\beta-\delta}(p - b) - c_2(s_2) = 0 \tag{A19}$$

Demand functions (A5-9) become:

$$d_{11i} = D_1 s_1 (n_1 s_1 + n_2 s_2)^{\alpha-1} p_{1i}{}^{-\delta} p^{\beta} g_1(t) \tag{A20}$$

$$d_{12i} = D_2 s_1 (n_1 s_1 + n_2 s_2)^{\alpha-1} p_{1i}{}^{-\delta} p^{\beta} g_2(t) \tag{A21}$$

$$d_{21i} = D_1 s_2 (n_1 s_1 + n_2 s_2)^{\alpha-1} p_{2i}{}^{-\delta} p^{\beta} g_1(t) \tag{A22}$$

$$d_{22i} = D_2 s_2 (n_1 s_1 + n_2 s_2)^{\alpha-1} p_{2i}{}^{-\delta} p^{\beta} g_2(t) \tag{A23}$$

Consumer surplus (9) an (10) become:

$$cs_1 = D_1 (n_1 s_1 + n_2 s_2)^{\alpha} \frac{b^{-\delta+1}}{\delta-1} p^{\beta} G_1(\infty) - D_1 (n_1 s_1 + n_2 s_2)^{\alpha} \frac{b^{-\delta+1} - p^{-\delta+1}}{\delta-1} p^{\beta} G_1(T_1) \tag{A24}$$

$$cs_2 = D_2 (n_1 s_1 + n_2 s_2)^{\alpha} \frac{b^{-\delta+1}}{\delta-1} p^{\beta} G_2(\infty) - D_2 (n_1 s_1 + n_2 s_2)^{\alpha} \frac{b^{-\delta+1} - p^{-\delta+1}}{\delta-1} p^{\beta} G_2(T_2) \tag{A25}$$

From (A16) an (A17):

$$n_1 s_1 + n_2 s_2 = \frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}} = \frac{\alpha-1}{\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}} \tag{A26}$$

Zero profit conditions (A18) and (A19) become:

$$[D_1G_1(T_1)+D_2G_2(T_2)]s_1\left(\frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}}\right)^{\alpha-1} p^{\beta-\delta}(p-b)-c_1(s_1)=0 \tag{A27}$$

$$[D_1G_1(T_1)+D_2G_2(T_2)]s_2\left(\frac{\alpha-1}{\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}}\right)^{\alpha-1} p^{\beta-\delta}(p-b)-c_2(s_2)=0 \tag{A28}$$

(A27-28) make s_1 and s_2 as functions of T_1 and T_2.

Consumer surpluses (A24-25) become:

$$cs_1=D_1\left(\frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}}\right)^{\alpha}\frac{b^{-\delta+1}}{\delta-1}p^{\beta}G_1(\infty)-D_1\left(\frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}}\right)^{\alpha}\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}G_1(T_1) \tag{A29}$$

$$cs_2=D_2\left(\frac{\alpha-1}{\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}}\right)^{\alpha}\frac{b^{-\delta+1}}{\delta-1}p^{\beta}G_2(\infty)-D_2\left(\frac{\alpha-1}{\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}}\right)^{\alpha}\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}G_2(T_2) \tag{A20}$$

First order conditions are:

$$\frac{dcs_1}{dT_1}=D_1\frac{b^{-\delta+1}}{\delta-1}p^{\beta}G_1(\infty)(\alpha-1)^{\alpha}(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}$$

$$-D_1\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}(\alpha-1)^{\alpha}(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}G_1(T_1)-$$

$$D_1\left(\frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}}\right)^{\alpha}\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}g_1(T_1)e^{-\gamma T_1}=0 \tag{A31}$$

$$\frac{dcs_2}{dT_2}=D_2\frac{b^{-\delta+1}}{\delta-1}p^{\beta}G_2(\infty)(\alpha-1)^{\alpha}(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_2}$$

$$-D_1\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}(\alpha-1)^{\alpha}(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_2}G_2(T_2)-$$

$$D_2\left(\frac{\alpha-1}{\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}}\right)^{\alpha}\frac{b^{-\delta+1}-p^{-\delta+1}}{\delta-1}p^{\beta}g_2(T_2)e^{-\gamma T_2}=0 \tag{A32}$$

We need the derivatives of s_1 and s_2 to T_1 and T_2. Take derivate to T_1 on both sides of (A27):

$$[D_1g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{\alpha-1}{\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}}\right)^{\alpha-1} p^{\beta-\delta}(p-b)+$$

$$[D_1G(T_1)+D_2G(T_2)]p^{\beta-\delta}(p-b)(\alpha-1)^{\alpha-1}\times$$

$$\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]\frac{\partial s_1}{\partial T_1}$$

$$-c_1{}'(s_1)\frac{\partial s_1}{\partial T_1}=0 \quad \text{(A33)}$$

Thus,

$$\frac{\partial s_1}{\partial T_1}=\frac{-[D_1 g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)} \quad \text{(A34)}$$

where

$$A \equiv p^{\beta-\delta}(p-b)(\alpha-1)^{\alpha-1} \quad \text{(A35)}$$

Similarly, take derivative to T_2 on both sides of (A28), one can find,

$$\frac{\partial s_2}{\partial T_2}=\frac{-[D_2 g_2(T_2)e^{-\gamma T_2}]s_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{1-\alpha}+s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\right]-c_2{}'(s_2)} \quad \text{(A36)}$$

Take derivative to T_2 on both sides of (A27),

$$[D_2 g_2(T_2)e^{-\gamma T_2}]As_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+$$

$$[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]\frac{\partial s_1}{\partial T_2}-$$

$$c_1{}'(s_1)\frac{\partial s_1}{\partial T_2}=0 \quad \text{(A37)}$$

$$\frac{\partial s_1}{\partial T_2}=\frac{-[D_2 g_2(T_2)e^{-\gamma T_2}]As_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}}{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'} \quad \text{(A38)}$$

Similarly,

$$\frac{\partial s_2}{\partial T_1}=\frac{-[D_1 g_1(T_1)e^{-\gamma T_1}]As_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}}{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}+s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\right]-c_2{}'} \quad \text{(A39)}$$

In addition, from (A18) and (A19), one has

$$\frac{c_1(s_1)}{s_1}=\frac{c_2(s_2)}{s_2} \quad \text{(A40)}$$

(A27), (A28), (A31), and (A32) are only equations of T_1, T_2, s_1, and s_2, given the derivatives in (A34), (A36), (A38), and (A39). (A27), (A28), (A31), and (A32) are collected here as follows:

$$f_1 = A[D_1 G_1(T_1) + D_2 G_2(T_2)] s_1 \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha} - c_1(s_1) = 0 \quad \text{(A41)}$$

$$f_2 = A[D_1 G_1(T_1) + D_2 G_2(T_2)] s_2 \left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} - c_2(s_2) = 0 \quad \text{(A42)}$$

$$f_3 = E_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1) - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1) - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1} G_1(T_1)$$

$$-F_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha} g_1(T_1) e^{-\gamma T_1} = 0 \quad \text{(A43)}$$

$$f_4 = E_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2) - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_2}$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2) - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_2} G_2(T_2)$$

$$-F_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha} g_2(T_2) e^{-\gamma T_2} = 0 \quad \text{(A44)}$$

Where

$$A \equiv p^{\beta-\delta}(p-b)(\alpha-1)^{\alpha-1} \quad \text{(A45)}$$

$$E_1 \equiv D_1 \frac{b^{-\delta+1}}{\delta-1} p^{\beta} G_1(\infty)(\alpha-1)^{\alpha} \quad \text{(A46)}$$

$$E_2 \equiv D_2 \frac{b^{-\delta+1}}{\delta-1} p^{\beta} G_2(\infty)(\alpha-1)^{\alpha} \quad \text{(A47)}$$

$$F_1 \equiv D_1 \frac{b^{-\delta+1} - p^{-\delta+1}}{\delta-1} p^{\beta} (\alpha-1)^{\alpha} \quad \text{(A48)}$$

$$F_2 \equiv D_2 \frac{b^{-\delta+1} - p^{-\delta+1}}{\delta-1} p^{\beta} (\alpha-1)^{\alpha} \quad \text{(A49)}$$

Newton's Method requires derivatives of f_1,f_2, f_3, and f_4 to s_1, s_2, T_1, and T_2.

$$\frac{\partial f_1}{\partial s_1} = A[D_1 G_1(T_1) + D_2 G_2(T_2)]\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)](1-\alpha)s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)-c_1{}'(s_1) \quad \text{(A50)}$$

$$\frac{\partial f_1}{\partial s_2}=0 \quad \text{(A51)}$$

$$\frac{\partial f_1}{\partial T_1}=A[D_1g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}\frac{\partial s_1}{\partial T_1}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}$$

$$-c'_1(s_1)\frac{\partial s_1}{\partial T_1} \quad \text{(A52)}$$

$$\frac{\partial f_1}{\partial T_2}=A[D_2g_2(T_2)e^{-\gamma T_2}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}\frac{\partial s_1}{\partial T_2}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]s_1(1-\alpha)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_2}-c'_1(s_1)\frac{\partial s_1}{\partial T_2} \quad \text{(A53)}$$

$$\frac{\partial f_2}{\partial s_1}=0 \quad \text{(A54)}$$

$$\frac{\partial f_2}{\partial s_2}=A[D_1G_1(T_1)+D_2G_2(T_2)]\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)-c_2{}'(s_2) \quad \text{(A55)}$$

$$\frac{\partial f_2}{\partial T_1}=A[D_1g_1(T_1)e^{-\gamma T_1}]s_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}\frac{\partial s_2}{\partial T_1}$$

$$+A[D_1G_1(T_1)+D_2G_2(T_2)]s_2(1-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_1}-c_2{}'\frac{\partial s_2}{\partial T_1} \quad \text{(A56)}$$

$$\frac{\partial f_2}{\partial T_2} = A[D_2 g_2(T_2) e^{-\gamma T_2}] s_2 \left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha}$$

$$+A[D_1 G_1(T_1) + D_2 G_2(T_2)] \left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} \frac{\partial s_2}{\partial T_2}$$

$$+A[D_1 G_1(T_1) + D_2 G_2(T_2)] s_2 (1-\alpha) \left(\frac{c_2{}'}{c_2} - \frac{1}{s_2}\right)^{-\alpha} \left(\frac{c_2{}'' c_2 - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right) \frac{\partial s_2}{\partial T_2} - c_2{}'(s_2) \frac{\partial s_2}{\partial T_2} \qquad \text{(A57)}$$

$$\frac{\partial f_3}{\partial s_1} = E_1(-\alpha)(-\alpha-1) \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-2} \left(\frac{c_1{}''(s_1) c_1(s_1) - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)^2 \frac{\partial s_1}{\partial T_1}$$

$$+E_1(-\alpha) \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}''' c_1 + c_1{}'' c_1{}' - 2c_1{}' c_1{}''}{c_1{}^2(s_1)} - 2c_1{}' \frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^3(s_1)} - 2\frac{1}{s_1{}^3}\right) \frac{\partial s_1}{\partial T_1}$$

$$+E_1(-\alpha) \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}''(s_1) c_1(s_1) - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right) \frac{\partial^2 s_1}{\partial T_1 \partial s_1}$$

$$-F_1(-\alpha)(-\alpha-1) \left(\frac{c_1{}'}{c_1} - \frac{1}{s_1}\right)^{-\alpha-2} \left(\frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^2} + \frac{1}{s_1{}^2}\right)^2 \frac{\partial s_1}{\partial T_1} G_1(T_1)$$

$$-F_1(-\alpha) \left(\frac{c_1{}'}{c_1} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}''' c_1 + c_1{}'' c_1{}' - 2c_1{}' c_1{}''}{c_1{}^2(s_1)} - 2c_1{}' \frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^3(s_1)} - 2\frac{1}{s_1{}^3}\right) \frac{\partial s_1}{\partial T_1} G_1(T_1)$$

$$-F_1(-\alpha) \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}''(s_1) c_1(s_1) - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right) \frac{\partial^2 s_1}{\partial T_1 \partial s_1} G_1(T_1)$$

$$-F_1(-\alpha) \left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^2} + \frac{1}{s_1{}^2}\right) g_1(T_1) e^{-\gamma T_1} \qquad \text{(A58)}$$

$$\frac{\partial f_3}{\partial s_2} = 0 \qquad \text{(A59)}$$

$$\frac{\partial f_3}{\partial T_1} = E_1(-\alpha)(-\alpha-1) \left(\frac{c_1{}'}{c_1} - \frac{1}{s_1}\right)^{-\alpha-2} \left(\frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)^2 \frac{\partial s_1}{\partial T_1} \frac{\partial s_1}{\partial T_1}$$

$$+E_1(-\alpha) \left(\frac{c_1{}'}{c_1} - \frac{1}{s_1}\right)^{-\alpha-1} \left(\frac{c_1{}''' c_1 + c_1{}'' c_1{}' - 2c_1{}' c_1{}''}{c_1{}^2(s_1)} - 2c_1{}' \frac{c_1{}'' c_1 - c_1{}'^2}{c_1{}^3(s_1)} - 2\frac{1}{s_1{}^3}\right) \frac{\partial s_1}{\partial T_1} \frac{\partial s_1}{\partial T_1}$$

$$+E_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial^2 s_1}{\partial T_1{}^2}$$

$$-F_1(-\alpha)(-\alpha-1)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-2}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2}+\frac{1}{s_1{}^2}\right)^2\left(\frac{\partial s_1}{\partial T_1}\right)^2 G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}'''c_1+c_1{}''c_1{}'-2c_1{}'c_1{}''}{c_1{}^2(s_1)}-2c_1{}'\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^3(s_1)}-2\frac{1}{s_1{}^3}\right)\left(\frac{\partial s_1}{\partial T_1}\right)^2 G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial^2 s_1}{\partial T_1{}^2}G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}g_1(T_1)e^{-\gamma T_1}$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_1}g_1(T_1)e^{-\gamma T_1}$$

$$-F_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left[g_1{}'(T_1)e^{-\gamma T_1}-\gamma g_1(T_1)e^{-\gamma T_1}\right] \quad (60)$$

$$\frac{\partial f_3}{\partial T_2}=E_1(-\alpha)(-\alpha-1)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-2}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)^2\frac{\partial s_1}{\partial T_1}\frac{\partial s_1}{\partial T_2}$$

$$+E_1(-\alpha)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}'''c_1+c_1{}''c_1{}'-2c_1{}'c_1{}''}{c_1{}^2(s_1)}-2c_1{}'\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^3(s_1)}-2\frac{1}{s_1{}^3}\right)\frac{\partial s_1}{\partial T_1}\frac{\partial s_1}{\partial T_2}$$

$$+E_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial^2 s_1}{\partial T_1\partial T_2}$$

$$-F_1(-\alpha)(-\alpha-1)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-2}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2}+\frac{1}{s_1{}^2}\right)^2\frac{\partial s_1}{\partial T_1}\frac{\partial s_1}{\partial T_2}G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'}{c_1}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}'''c_1+c_1{}''c_1{}'-2c_1{}'c_1{}''}{c_1{}^2(s_1)}-2c_1{}'\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^3(s_1)}-2\frac{1}{s_1{}^3}\right)\frac{\partial s_1}{\partial T_1}\frac{\partial s_1}{\partial T_2}G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''(s_1)c_1(s_1)-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial^2 s_1}{\partial T_1\partial T_2}G_1(T_1)$$

$$-F_1(-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\frac{\partial s_1}{\partial T_2}g_1(T_1)e^{-\gamma T_1} \tag{A61}$$

$$\frac{\partial f_4}{\partial s_1}=0 \tag{A62}$$

$$\frac{\partial f_4}{\partial s_2}=E_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2\partial s_2}$$

$$-F_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2\partial s_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)g_2(T_2)e^{-\gamma T_2} \tag{A63}$$

$$\frac{\partial f_4}{\partial T_1}=E_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_1}\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_1}\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2\partial T_1}$$

$$-F_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_1}\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_1}\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2\partial T_1}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_1}g_2(T_2)e^{-\gamma T_2} \quad \text{(A64)}$$

$$\frac{\partial f_4}{\partial T_2}=E_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_2}\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_2}\frac{\partial s_2}{\partial T_2}$$

$$+E_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2{}^2}$$

$$-F_2(-\alpha)(-\alpha-1)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-2}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)^2\frac{\partial s_2}{\partial T_2}\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'}{c_2}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}'''c_2+c_2{}''c_2{}'-2c_2{}'c_2{}''}{c_2{}^2(s_2)}-2c_2{}'\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^3(s_2)}-2\frac{1}{s_2{}^3}\right)\frac{\partial s_2}{\partial T_2}\frac{\partial s_2}{\partial T_2}G_2(T_2)$$

$$-F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''(s_2)c_2(s_2)-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\frac{\partial^2 s_2}{\partial T_2{}^2}G_2(T_2)$$

$$-2F_2(-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2}+\frac{1}{s_2{}^2}\right)\frac{\partial s_2}{\partial T_2}g_2(T_2)e^{-\gamma T_2}$$

$$-F_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(g'_2(T_2)e^{-\gamma T_2}-\gamma g_2(T_2)e^{-\gamma T_2}\right) \quad \text{(A65)}$$

We need the second order derivatives of s_1 and s_2 to T_1 and T_2.

$$\frac{\partial^2 s_1}{\partial T_1{}^2}=$$

$$\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A-[D_1g_1(T_1)e^{-\gamma T_1}](1-\alpha)s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)}\frac{\partial s_1}{\partial T_1}$$

$$
+\frac{-[D_1(g_1{}'(T_1)-\gamma g_1(T_1))e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)}
$$

$$
-\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A}{\left\{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)\right\}^2}
$$

$$
\times\left\{\begin{matrix}[D_1g_1(T_1)e^{-\gamma T_1}]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]\\ +[D_1G_1(T_1)+D_2G_2(T_2)]A[H]\frac{\partial s_1}{\partial T_1}-c_1{}''(s_1)\frac{\partial s_1}{\partial T_1}\end{matrix}\right\} \quad \text{(A66)}
$$

$$
\frac{\partial^2 s_1}{\partial T_1\partial T_2}=
$$

$$
\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A-[D_1g_1(T_1)e^{-\gamma T_1}](1-\alpha)s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)}\frac{\partial s_1}{\partial T_2}
$$

$$
-\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}A}{\left\{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'(s_1)\right\}^2}
$$

$$
\times\left\{\begin{matrix}[D_2g_2(T_2)e^{-\gamma T_2}]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]\\ +[D_1G_1(T_1)+D_2G_2(T_2)]A[H]\frac{\partial s_1}{\partial T_2}-c_1{}''(s_1)\frac{\partial s_1}{\partial T_2}\end{matrix}\right\} \quad \text{(A67)}
$$

Where

$$
H\equiv 2(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)
$$

$$
+(1-\alpha)(-\alpha)s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha-1}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)^2
$$

$$
+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}'''c_1+c_1{}''c_1{}'-2c_1{}'c_1{}''}{c_1{}^2(s_1)}-2c_1{}'\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^3(s_1)}-2\frac{1}{s_1{}^3}\right) \quad \text{(A68)}
$$

$$\frac{\partial^2 s_2}{\partial {T_2}^2} =$$

$$\frac{-D_2\left[g'_2(T_2)e^{-\gamma T_2} - \gamma g_2(T_2)e^{-\gamma T_2}\right] s_2 \left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} A}{[D_1G_1(T_1) + D_2G_2(T_2)]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] - {c_2}'(s_2)}$$

$$+\frac{-[D_2g_2(T_2)e^{-\gamma T_2}]\left[\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] A\frac{\partial s_2}{\partial T_2}}{[D_1G_1(T_1) + D_2G_2(T_2)]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] - {c_2}'(s_2)}$$

$$-\frac{-[D_2g_2(T_2)e^{-\gamma T_2}] s_2\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} A}{\left\{[D_1G_1(T_1) + D_2G_2(T_2)]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] - {c_2}'(s_2)\right\}^2}$$

$$\times\left\{\begin{matrix}[D_2g_2(T_2)e^{-\gamma T_2}]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] \\ +[D_1G_1(T_1) + D_2G_2(T_2)]AJ\frac{\partial s_2}{\partial T_2} - {c_2}''\frac{\partial s_2}{\partial T_2}\end{matrix}\right\} \quad \text{(A69)}$$

$$J \equiv 2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)$$

$$+s_2(1-\alpha)(-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha-1}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)^2$$

$$+s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}'''c_2 + {c_2}''{c_2}' - 2{c_2}'{c_2}''}{{c_2}^2(s_2)} - 2{c_2}'\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^3(s_2)} - 2\frac{1}{{s_2}^3}\right) \quad \text{(A70)}$$

$$\frac{\partial^2 s_2}{\partial T_2 \partial T_1} =$$

$$+\frac{-[D_2g_2(T_2)e^{-\gamma T_2}]\left[\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] A\frac{\partial s_2}{\partial T_1}}{[D_1G_1(T_1) + D_2G_2(T_2)]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] - {c_2}'(s_2)}$$

$$-\frac{-[D_2g_2(T_2)e^{-\gamma T_2}] s_2\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} A}{\left\{[D_1G_1(T_1) + D_2G_2(T_2)]A\left[\left(\frac{{c_2}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{{c_2}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{{c_2}''c_2 - {c_2}'^2}{{c_2}^2(s_2)} + \frac{1}{{s_2}^2}\right)\right] - {c_2}'(s_2)\right\}^2}$$

$$\times \left\{ \begin{array}{c} [D_1 g_1(T_1) e^{-\gamma T_1}] A \left[\left(\frac{c_2{}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2 - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)\right] \\ +[D_1G_1(T_1)+D_2G_2(T_2)]AJ\frac{\partial s_2}{\partial T_1} - c_2{}''\frac{\partial s_2}{\partial T_1} \end{array} \right\} \tag{A71}$$

$$\frac{\partial^2 s_1}{\partial T_1 \partial s_1} =$$

$$\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha} A - [D_1g_1(T_1)e^{-\gamma T_1}]s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1 - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha} + s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1 - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)\right] - c_1{}'(s_1)}$$

$$- \frac{-[D_1g_1(T_1)e^{-\gamma T_1}]s_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha} A}{\left\{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{1-\alpha} + s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)} - \frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1 - c_1{}'^2}{c_1{}^2(s_1)} + \frac{1}{s_1{}^2}\right)\right] - c_1{}'(s_1)\right\}^2}$$

$$\times \left\{[D_1G_1(T_1)+D_2G_2(T_2)]A[H] - c_1{}''(s_1)\right\} \tag{A72}$$

$$\frac{\partial^2 s_1}{\partial T_1 \partial s_2} = 0 \tag{A73}$$

$$\frac{\partial^2 s_2}{\partial T_2 \partial s_2} =$$

$$\frac{-[D_2g_2(T_2)e^{-\gamma T_2}]\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} A - [D_2g_2(T_2)e^{-\gamma T_2}]s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2 - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)A}{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_2{}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2 - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)\right] - c_2{}'(s_2)}$$

$$- \frac{-[D_2g_2(T_2)e^{-\gamma T_2}]s_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{1-\alpha} A}{\left\{[D_1G_1(T_1)+D_2G_2(T_2)]A\left[\left(\frac{c_2{}'}{c_2} - \frac{1}{s_2}\right)^{1-\alpha} + s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)} - \frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2 - c_2{}'^2}{c_2{}^2(s_2)} + \frac{1}{s_2{}^2}\right)\right] - c_2{}'(s_2)\right\}^2}$$

$$\times [D_1G_1(T_1)+D_2G_2(T_2)]AJ - c_2{}''(s_2) \tag{A74}$$

$$\frac{\partial^2 s_1}{\partial T_2 \partial s_1} =$$

$$\frac{-[D_2g_2(T_2)e^{-\gamma T_2}]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]}{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'}$$

$$-\frac{-[D_2g_2(T_2)e^{-\gamma T_2}]As_1\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}}{\left\{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{1-\alpha}+s_1(1-\alpha)\left(\frac{c_1{}'(s_1)}{c_1(s_1)}-\frac{1}{s_1}\right)^{-\alpha}\left(\frac{c_1{}''c_1-c_1{}'^2}{c_1{}^2(s_1)}+\frac{1}{s_1{}^2}\right)\right]-c_1{}'\right\}^2}$$

$$\times\left\{[D_1G(T_1)+D_2G(T_2)]AH-c_1{}''\right\} \qquad \text{(A75)}$$

$$\frac{\partial^2 s_2}{\partial T_1\partial s_2}=$$

$$\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]A\left[\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}+s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\right]}{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}+s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\right]-c_2{}'}$$

$$-\frac{-[D_1g_1(T_1)e^{-\gamma T_1}]As_2\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}}{\left\{[D_1G(T_1)+D_2G(T_2)]A\left[\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{1-\alpha}+s_2(1-\alpha)\left(\frac{c_2{}'(s_2)}{c_2(s_2)}-\frac{1}{s_2}\right)^{-\alpha}\left(\frac{c_2{}''c_2-c_2{}'^2}{c_2{}^2(s_2)}+\frac{1}{s_2{}^2}\right)\right]-c_2{}'\right\}^2}$$

$$\times\left\{[D_1G(T_1)+D_2G(T_2)]AJ-c_2{}''\right\} \qquad \text{(A76)}$$